Impact of the Number of Neutrals on Stability Concepts in Friends Oriented Hedonic Games

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Abstract

We study hedonic games under friends appreciation, where each agent considers other agents friends, enemies, or neutrals. Although existing work assumed that neutrals have no impact on an agent's preference, it may be that her preference depends on the number of neutrals in her coalition. We extend the existing preference by allowing an agent to take into consideration the number of neutrals in her coalition. Even though the impact of neutrals on preference is negligible compared to the impact of friends and enemies, it does affect the stability of coalition structures. When each agent prefers coalitions with more neutrals, we show that both core stable outcomes and individually stable outcomes might not exist. We also prove that deciding the existence of the core is NP^{NP}-complete, whereas deciding the existence of an individual stable coalition is NP-complete.

1 Introduction

In many real-life examples, ranging from sports clubs to political parties, individuals (*agents*) carry out activities in groups (*coalitions*) to achieve common goals. In coalition formation with hedonic preferences, or *hedonic games*, each agent's payoff only depends on the coalition that she joins. A natural question is then whether a stable partition of the set of agents (*coalition structure*) exists. Various stability concepts have been studied in the literature; among the most prominent of them are *core* and *individual stability* (see handbook chapter [Aziz and Savani, 2016]).

Since the number of coalitions that an agent can join is exponential, several compact preference representations of hedonic games have been proposed. In particular, Dimitrov et al. [2006] developed simple hedonic games where an agent divides the other agents into friends or enemies. They studied a preference called *friends appreciation* where each agent prefers coalitions with more friends and, in case of a tie, with fewer enemies. More recently, Ohta et al. [2017] proposed a slight extension of this model by introducing the existence of neutrals who do not impact the preference. Under friends appreciation, in the former model, an individual stable coalition structure always exists [Dimitrov and Sung, 2004], whereas, in the latter, the question has yet to be studied. In both models under friends appreciation, there always exists a core stable coalition structure, which can be found in polynomial time.[1]

We propose a natural extension of friends appreciation when friends, enemies, and neutrals exist. In our new model, each agent takes into account the number of neutrals in her coalition, i.e., the existence of neutrals can be either slightly positive or negative. In this paper, we assume a sociable agent slightly prefers coalitions with more neutrals, while an introverted agent slightly prefers coalitions with fewer neutrals. In this model, the impact of neutrals on an agent's preference is almost

[1] There exists an alternative preference called *enemies aversion*, where each agent prefers coalitions with fewer enemies and in case of a tie, with more friends. Under enemies aversion, without neutrals, there always exists a core outcome, but with neutrals, the core can be empty [Ohta et al., 2017].

negligible compared to that of a friend or an enemy. From a theoretical viewpoint, analyzing how neutral agents affect stability concepts more precisely is meaningful. Since in a more general model with additively separable preferences, a core/individual stable outcome may not exist, we hope to clarify a boundary case where core/individual stability is guaranteed.

Our results show that the number of neutral agents impacts the existence of the core and its computational complexity. The existence of a core stable coalition structure is not guaranteed anymore for sociable agents. We also study individual stability under the original friends appreciation and our extended preference, showing that an individual stable coalition structure may not exist. Furthermore, for both stability notions, under friends appreciation with sociable agents, we investigate the complexity of (VERIF) to verify whether a given coalition structure is stable and (EXIST) to decide the existence of a stable coalition structure. In particular, we show that deciding whether the core is empty is NP^{NP}-complete, whereas deciding whether an individual stable coalition structure exists is NP-complete.

Related work In hedonic games, initiated by Banerjee et al. [2001] and Bogomolnaia and Jackson [2002], a fundamental question is to identify the necessary and sufficient conditions on preferences for the existence of stable coalition structure. More recently, Aziz and Brandl [2012] clarified the relationships among stability concepts such as core, Nash or individual stability, and provided some existence results. In fractional hedonic games, Aziz et al. [2014] described the conditions that guarantee the existence of a core stable outcome, and Brandl et al. [2015] showed that an individual stable outcome may fail to exist. Aziz et al. [2016] proposed Boolean hedonic games where each agent partitions the set of other agents into satisfactory and unsatisfactory groups and showed core non-emptiness. Lang et al. [2015] introduced the idea of neutral agents and characterized the coalition structures that necessarily/possibly satisfy some stability concepts. Furthermore, Peters [2016] proposed a graphical representation of hedonic games, where an agent's utility only depends on her neighbors' actions.

Regarding computational complexity, many results exist for verification and existence problems for various stability concepts. Ballester [2004] showed that the existence problem is NP-complete for core and individual stability, under individually rational coalition lists. In additively separable hedonic games, Woeginger [2013] showed that the existence problem for the core is NP^{NP}-complete. Peters and Elkind [2015] developed a framework to prove NP-hardness of existence problems, which applies to various hedonic games such as hedonic coalition nets [Elkind and Wooldridge, 2009].

Outline In Section 2, we present our hedonic game model and stability concepts. In Section 3, we show that the core may be empty and we examine the complexity of deciding its existence. In Section 4, we provide counter-examples for individual stability, and, with sociable agents, we study the complexity of the existence problem. Finally, in Section 5, we discuss stability in the presence of introverted agents.

2 Preliminaries

Let $N = \{1, \dots, n\}$ denote the *set of agents*. A *coalition* $C \subseteq N$ is a subset of agents. A *coalition structure* π is a partition of N. Let $\pi(i)$ denote the coalition to which agent i belongs in π. Let C^N denote the set of all coalition structures. For every agent i, her *preference* $\succsim_i$ is based on the coalitions to which she belongs; let $\succ_i$ and $\sim_i$ respectively denote the strict preference and the indifference relation derived from $\succsim_i$.

A *hedonic game* (N, P) is defined by set of agents N and *preference profile* $P = (\succsim_i)_{i \in N}$. *Additively separable hedonic games* form a natural class of hedonic games where each agent has a value for any other agent and the utility that an agent derives from a coalition is the sum of the values that she has for its members.

Definition 1 (Additively Separable). *A hedonic game* $(N, \succsim)$ *is* additively separable *if for each agent* $i \in N$ *there exists a utility function* $v_i : N \mapsto \mathbb{R}$ *such that* $v(i) = 0$ *and for any two coalitions*

S, T *such that* $i \in S, T$,

$$S \succsim_i T \Leftrightarrow \sum_{j \in S} v_i(j) \geq \sum_{j \in T} v_i(j).$$

All the games that we consider in this paper are additively separable. Furthermore, an additively separable hedonic game is *symmetric* when any two agents associate the same value to each other.

Definition 2 (Symmetry). *An additively separable hedonic game satisfies* symmetry *if for all* $i, j \in N$, $v_i(j) = v_j(i)$.

We concentrate on two prominent stability concepts, *core* and *individual* stability. These stability concepts are among the least restrictive ones -after *individual rationality*- which concern respectively coalition and individual deviations. Individual rationality is a minimal stability requirement which guarantees that each player weakly prefers her coalition over being alone.

Definition 3 (Individual Rationality). *A coalition structure* $\pi \in C^N$ *is individually rational if there exists no agent* $i \in N$ *that has an incentive to deviate alone, i.e.,* $\{i\} \succ_i \pi(i)$.

Definition 4 (Core Stability). *A coalition structure* $\pi \in C^N$ *admits a blocking coalition* $X \subseteq N$ *(*$X \neq \emptyset$*) if for every* $i \in X$, $X \succ_i \pi(i)$ *holds. The core is the set of coalition structures that do not admit any blocking coalitions.*

Definition 5 (Individual Stability). *A coalition structure* $\pi \in C^N$ *is individually stable if there exists no pair of agent* $i \in N$ *and coalition* $C \in \pi \cup \{\emptyset\}$ *such that* $C \cup \{i\} \succ_i \pi(i)$, *and* $C \cup \{i\} \succsim_j C$ *for all* $j \in C$.

Intuitively, a coalition structure is core stable if no group of agents benefits from forming a deviating coalition, and it is individually stable if no individual agent benefits from joining an existing coalition, without harming any agent in this coalition. Furthermore, a coalition C is *acceptable* to agent i if and only if $C \succsim_i \{i\}$ holds. Thus if $\pi(i)$ is unacceptable to agent i, π cannot be a member of the core or individually stable.

We consider a simple and compact preference called *friends appreciation*. For each agent i, set N is partitioned into $\{F_i, \perp_i, E_i\}$. Agents F_i are her *friends*, E_i are her *enemies*, and $\perp_i$ are *neutral* agents.

Let us define the original *friends appreciation* proposed by Dimitrov et al. [2006]. When comparing two coalitions under friends appreciation, an agent first compares the number of her friends in each coalition, and then the number of her enemies. For two coalitions C and D, agent i prefers the coalition with more friends, and in case of a tie, she prefers the one with fewer enemies:

$$C \succ_i D \Leftrightarrow \begin{cases} |C \cap F_i| > |D \cap F_i| \text{ or} \\ |C \cap F_i| = |D \cap F_i| \text{ and } |C \cap E_i| < |D \cap E_i|. \end{cases}$$

Furthermore, $C \sim_i D$ holds iff $|C \cap F_i| = |D \cap F_i|$ and $|C \cap E_i| = |D \cap E_i|$. The set of preference profiles under friends appreciation is denoted by $\mathcal{P}^F$. Note that a preference in $\mathcal{P}^F$ is additively separable with weights n for a friend, 0 for a neutral, and -1 for an ennemy.

We slightly generalize friends appreciation by allowing agents to take into account in their preferences the number of neutrals (after having considered the number of their friends and their enemies). We propose two alternative preferences, *friends appreciation with sociable agents* and *friends appreciation with introverted agents*, depending on whether agents believe that neutrals have a positive or a negative impact. Under friends appreciation with sociable agents, in the case of a tie for the number of both friends and enemies, agent i prefers the coalition with more neutrals, i.e., for two coalitions, C and D:

$$C \succ_i D \Leftrightarrow \begin{cases} |C \cap F_i| > |D \cap F_i|, \text{ or} \\ |C \cap F_i| = |D \cap F_i| \text{ and } |C \cap E_i| < |D \cap E_i|, \text{ or} \\ |C \cap F_i| = |D \cap F_i| \text{ and } |C \cap E_i| = |D \cap E_i| \text{ and } |C \cap \perp_i| > |D \cap \perp_i|. \end{cases}$$

Similarly, under friends appreciation with introverted agents, agent i prefers the coalition with fewer neutrals. For both preferences, $C \sim_i D$ holds iff $|C \cap F_i| = |D \cap F_i|$, $|C \cap E_i| = |D \cap E_i|$, and $|C \cap \perp_i| = |D \cap \perp_i|$. The set of preference profiles under friends appreciation with sociable agents (resp. introverted agents) is denoted by $\mathcal{P}^{F+}$ (resp. $\mathcal{P}^{F-}$). A preference in $\mathcal{P}^{F+}$ (resp. $\mathcal{P}^{F-}$) is additively separable with weights n for a friend, ϵ for a neutral, and -1 for an ennemy (resp. n,$-\epsilon$, and -1), for $0 < \epsilon << 1$.

Definition 6 (HG/F, HG/F+, and HG/F-)**.** *An* HG/F *(resp.* HG/F+, HG/F-*) is a hedonic game $(N, (\succsim_i)_{i \in N})$ such that each $\succsim_i$ is in $\mathcal{P}^F$ (resp. $\mathcal{P}^{F+}$, $\mathcal{P}^{F-}$).*

Hedonic games applying these agents' preferences can be represented by a labeled digraph, $G_{EF\perp} = (N, A_E \cup A_F \cup A_\perp)$, where each vertex represents an agent, and arc (i, j) in set A_E labeled by E (resp. by $F, \perp$) indicates that agent i considers agent j an enemy (resp. friend, neutral).

In the context of hedonic games, two well-studied decision problems are Existence and Verification. Given a stability concept and a hedonic game, the former decides whether there exists a stable coalition structure, and the latter verifies whether a given coalition structure is stable. For HG/F+, we define HG/F+/IS/EXIST and HG/F+/C/EXIST as the existence problems related to individual and core stability. Similarly, we define HG/F+/IS/VERIF and HG/F+/C/VERIF as the verification problems. We assume the reader is familiar with concepts from complexity theory, particularly with time complexity classes NP, NP^{NP}, and their complements [Garey and Johnson, 2002]. In our complexity proofs, we utilize the NP-complete problem MAXCLIQUE and the coNP^{NP}-complete problem MINMAXCLIQUE [Ko and Lin, 1995], defined below.

Definition 7 (Problem MAXCLIQUE)**.** *Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ and a lower threshold $k \in \mathbb{N}$. Does it exist a subset of k vertices $\mathcal{W} \subseteq \mathcal{V}$ such that subgraph $\mathcal{G}[\mathcal{W}]$ is a clique?*[2]

Definition 8 (Problem MINMAXCLIQUE[3])**.** *Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, two sets I, J that partition $\mathcal{V}$ into $\{\mathcal{V}_{i,j} \mid i \in I, j \in J\}$, and a lower threshold $k \in \mathbb{N}$. For every function $t : I \to J$, does the subgraph $\mathcal{G}[\cup_{i \in I} \mathcal{V}_{i,t(i)}]$ contain a clique of size k?*

In other words, $\mathcal{V}$ is partitioned into $|I| \cdot |J|$ subsets $\mathcal{V}_{i,j}$. Then, for each function $t : I \to J$, we consider a MAXCLIQUE problem for the subgraph induced by $\mathcal{G}[\cup_{i \in I} \mathcal{V}_{i,t(i)}]$.

3 Core Stability with Sociable Agents

In this section, we discuss the existence of a core stable coalition structure under friends appreciation with sociable agents, as well as the complexity of the existence problem.

3.1 The core may be empty

When neutral agents have no impact on preferences, Ohta et al. [2017] showed that a core stable coalition structure always exists and that it can be computed in polynomial time as the strongly connected components of graph $G_F = (N, A_F)$. These results also hold in the original friends and enemies model (without neutral agents) [Dimitrov et al., 2006]. However, when all agents prefer coalitions with more neutrals, the existence of a core stable coalition structure is no longer guaranteed.

Theorem 1. *In an HG/F+, the core may be empty.*

To prove this theorem, we utilize the following example:

[2] A graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ is a clique if and only if for any pair of nodes x, y in $\mathcal{V}$, edge (x, y) belongs to $\mathcal{A}$.

[3] The complexity proof from Ko and Lin [1995] even holds when $J = \{0, 1\}$ and $|\mathcal{V}_{i,0}| = |\mathcal{V}_{i,1}|$ for every $i \in I$.

Example 1. *Assume there exist six agents:* $\{1, 2, 3, 4, 5, 5'\}$*. First, for* $i \in \{1, 2, 3\}$*, agent* i *considers agent* $i + 1$ *a friend, whereas* $i + 1$ *regards* i *as neutral. Agents* 4*,* 5*, and* $5'$ *view each other as neutral. Agents* 5 *and* $5'$ *view agent* 1 *as a friend, but* 1 *considers them as neutral. All other relations are enemy relations.*
We illustrate these preferences with Figure 1 which is a representation of Graph $G_{F\perp}$*.*

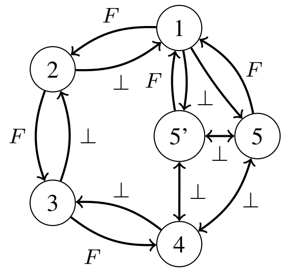

Figure 1: Example of an HG/F+ with an empty core.

Proof. By contradiction, assume that a core stable coalition structure π exists in Example 1.
First, assume that agents 5 and $5'$ belong to different coalitions in π. If agent 1 does not belong to $\pi(2)$, then coalition $\{1, 5, 5'\}$ is a deviation; thus $\pi(1) = \pi(2)$. If 5 belongs to $\pi(1)(= \pi(2))$, then 3 belongs to $\pi(1)$ (otherwise 2 deviates alone), which implies that 4 also belongs to $\pi(1)$ (otherwise 3 deviates alone). However, 4 has enemies but no friend in this coalition, thus 4 deviates alone. Therefore 5 does not belong to $\pi(1)$, and by symmetry between 5 and $5'$, $\{5, 5'\} \cap \pi(1) = \emptyset$. Now, if agent 4 does not belong to $\pi(3)$, coalition $\{4, 5, 5'\}$ is a deviation, but when 4 belongs to $\pi(3)$, coalition $\{5, 5'\}$ is a deviation. Therefore, 5 and $5'$ belong to the same coalition, and since they have identical preferences, we consider them as a single agent $5\text{-}5'$ in the following.
Assume that there exists a coalition that contains three agents or more from $\{1, 2, 3, 4, 5\text{-}5'\}$. This coalition must include an agent with no friend and at least one enemy, who then prefers to deviate in a singleton coalition. Therefore, each coalition consists of at most two agents, which implies that at least one agent is in a singleton coalition. However, if agent 1 (resp. 2, 3, 4) is in a singleton coalition, then coalition $\{1, 5\text{-}5'\}$ (resp. $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$) is a deviation, since agent $5\text{-}5'$ (resp. 1, 2, 3) gains one friend. Similarly, if agent $5\text{-}5'$ is in a singleton coalition, then coalition $\{4, 5\text{-}5'\}$ is a deviation since 4 gains one neutral. As a result, there is no core stable coalition structure in Example 1. □

3.2 Computational Complexity

In this subsection, we study the complexity of the existence of a core stable outcome under friends appreciation with sociable agents. First, notice that an HG/F+ where there exists no friend relation is equivalent to a hedonic game under *enemies aversion* (see Footnote 1) where the neutral arcs from the original graph become friend arcs in the second. Since under enemies aversion the complexity of verifying that a given coalition structure is in the core is coNP-complete [Sung and Dimitrov, 2007], it extends to our setting:

Theorem 2. *Problem* HG/F+/C/VERIF *is coNP-complete.*

This result implies that the corresponding existence problem is in NP^{NP}. Moreover, we show the following:

Theorem 3. *Problem* HG/F+/C/EXIST *is* $\textit{NP}^{\textit{NP}}$*-complete.*

Before presenting the proof, we present two useful remarks:

Remark 1. *Consider an HG/F+ where there exists a clique of friends, K, in which agents have no friends outside of K. Then a coalition structure π that divides the agents of K into different coalitions is not core stable (since K is a deviation for π).*

Remark 2. *Consider an HG/F+, $(N, (\succ_i)_{i \in N})$, composed of b cliques of friends, $N = \{K^1, \ldots, K^b\}$, and assume that 1) agents have no friend outside of their clique, and 2) agents in the same clique have the same set of neutrals, which is a subset of $\{K^1, \ldots, K^b\}$, and 3) the game is symmetric. Then, a core stable coalition exists.*
Indeed, core stability in this game is equivalent to core stability in hedonic game $(B = \{1, \ldots, b\}, (\succ_j)_{j \in B})$ under enemies aversion without neutrals (see Footnote 1), where each clique of friends, $K^j, j \in \{1, \ldots, b\}$, is represented by a single node j, and for all $j, j' \in \{1, \ldots, b\}, j' \in F_j$ (resp. $j' \in E_j$) if and only if for any $i \in K^j, i' \in K^{j'}, i' \in \perp_i$ (resp. $i' \in E_i$).

Proof of Theorem 3. The main argument of the proof resembles to the one of Theorem 4 in [Ohta et al., 2017] concerning HG/E, however we have to adapt it to our setting. Indeed, while in an HG/E coalitions in a core stable partition are necessarily cliques (in the graph $G_{F\perp}$), this does not hold in an HG/F+. We adapt it by introducing cliques where only single nodes were necessary in their proof, i.e., we introduce vertex-cliques, a fulcrum-clique, and cliques for inhibitors and logic games (all described below).

First, by Theorem 2, HG/F+/C/EXIST is in NP^{NP}. We prove NP^{NP}-hardness by reducing the coNP^{NP}-complete MINMAXCLIQUE to the complement of HG/F+/C/EXIST. Let an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, a set I partitioning set $\mathcal{V}$ into $\{\mathcal{V}_{i,0}, \mathcal{V}_{i,1} \mid i \in I\}$, and an integer $k \in \mathbb{N}$, define a restricted instance of MINMAXCLIQUE, where $\forall i \in I, |\mathcal{V}_{i,0}| = |\mathcal{V}_{i,1}|$ (see Footnote 3). We construct the corresponding instance of coHG/F+/C/EXIST, illustrated by a partial representation of $G_{F\perp}$ in Figure 2, as follows:
(1) For each x in $\mathcal{V}$, we create *vertex-clique* K^x that contains k' mutual friends (k' is specified at the end of the proof), and then $V = \bigcup_{i \in I} V_{i,0} \cup V_{i,1}$, where $V_{i,j} = \bigcup_{x \in \mathcal{V}_{i,j}} K^x$. For each edge (x, y) in graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, we introduce mutually neutral arcs between each $x' \in K^x$ and $y' \in K^y$.
(2) We introduce a *generalization of Example 1* where agents 5 and $5'$ are replaced by K^μ, a clique of $k'' - 1$ mutually neutral agents (k'' is specified at the end of the proof).
(3) We introduce a *fulcrum-clique* of 2 mutual friends, K^φ. Agents in K^φ consider agents in K^μ and in V neutral. Agents in K^μ view agents in K^φ as friends.
(4) Between each pair $V_{i,0}$ and $V_{i,1}$, we introduce $|\mathcal{V}_{i,0}|$ *inhibitors* (specified below). Each vertex-clique K^x in $V_{i,0}$ is paired to one vertex-clique K^y in $V_{i,1}$ through inhibitor H^x_y which makes exactly one of them available for a core stable coalition with K^φ.
(5) We connect to each $V_{i,j}$ a *logic game* $L_{i,j}$ (specified below) containing a blocking coalition iff condition 'all agents in $V_{i,j}$ are inhibited, or none is inhibited' is not satisfied.
(6) All others relations are enemies relations.

Before the main argument, notice that all the vertex-cliques $K^x, x \in \mathcal{V}$, the fulcrum-clique K^φ, the inhibitors (described below), and the cliques from the logic games (described below) satisfy Remark 1. Therefore we do not discuss the stability of the coalition structures that divide the agents of those cliques.

Main argument
First, observe that no core stable coalition structure contains a coalition with agents from the left and the right of K^φ (w.r.t. Figure 2). Thus, in a core stable coalition, K^φ is either grouped with K^μ (of size $k'' - 1$) or a clique whose size is at least k'' in V. If clique K^φ goes to the left, the game from the generalization of Example 1 is isolated and thus has an empty core. However, if clique K^φ goes to the right, the core of game $K^\varphi \cup$ Example 1 is non-empty: $\{\{K^\varphi, K^\mu\}, \{1, 2\}, \{3, 4\}\}$ is core stable.

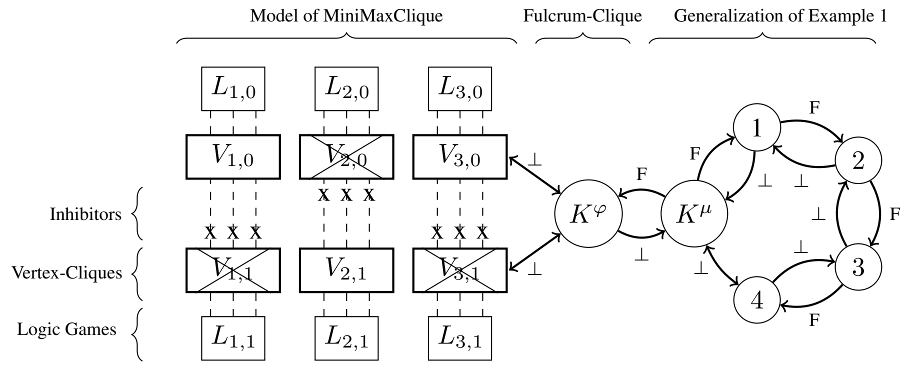

Figure 2: Instance of coHG/F+/C/EXIST from MINMAXCLIQUE.

Assume that there exists a function $t^* : I \to \{0, 1\}$ such that subgraph $\mathcal{G}[\cup_{i \in I} \mathcal{V}_{i,t^*(i)}]$ contains no clique of size k. We claim that a core stable coalition structure exists. For each $i \in I$, we set the inhibitors to $V_{i,1-t^*(i)}$. Thus no logic game generates a blocking coalition and only agents in $\cup_{i \in I} V_{i,t^*(i)}$ are available to K^{φ}. Thus, if we group cliques K^{φ} and K^{μ} together, K^{φ} has no interest to deviate left, which implies that partial coalition structure $\{\{K^{\varphi}, K^{\mu}\}, \{1, 2\}, \{3, 4\}\}$ admits no blocking coalition (in the whole game). Furthermore, the subgraph composed of vertex-cliques K^x in $\cup_{i \in I} V_{i,t^*(i)}$ represents a game that satisfies Remark 2, and thus, a core stable partition exists[4].

Conversely, assume that for every function $t : I \to \{0, 1\}$, subgraph $\mathcal{G}[\cup_{i \in I} \mathcal{V}_{i,t(i)}]$ contains a clique of size k. By contradiction, assume that a core stable coalition structure π exists. Then there exists function t^{π} such that, for each $i \in I$, all inhibitors between $V_{i,0}$ and $V_{i,1}$ are set to $1 - t^{\pi}(i)$, otherwise at least one logic game is not core stable. Thus K^{φ} is grouped with a clique which size is at least k'' in $G_{F\perp}[\cup_{i \in I} V_{i,t^{\pi}(i)}]$, which exists for any function $t : I \to \{0, 1\}$ by assumption.

Inhibitors and logic games

Inhibitors and logic games help us model the MINMAXCLIQUE problem; the former make agents (un)available for a deviation with K^{ϕ}, and the later impose that the set of available agents correspond to a function $t : I \to \{0, 1\}$.

An *inhibitor*, H_y^x, is a clique that contains k^* mutual friends (k^* is specified at the end of the proof) and pairs one vertex-clique K^x in $V_{i,0}$ with one vertex-clique K^y in $V_{i,1}$, such that each K^x in $V_{i,0}$ is paired to exactly one K^y in $V_{i,1}$ and conversely. Each agent in H_y^x is mutually neutral with each agent in K^x and K^y. Thus, inhibitor H_y^x either joins all the agents in K^x or all the agents in K^y (but not both or partially) in a core stable coalition. Furthermore, when H_y^x joins K^x (resp. K^y), we enforce that K^x (resp. K^y) prefers to stay with H_y^x over any other coalition by properly fixing k^*, the size of H_y^x.

Each *logic game* $L_{i,j}$ relies on a combination of gadget games, presented in Figure 3, that model logical gates with the understanding that an available agent amounts to Boolean True.

All K^* in Figure 3 are cliques of friends. In gate NOT, we assume that K^x and K^y have identical size, then the availability of K^x makes K^y non-available. In gate OR, we assume that K^{x_1} and K^{x_2} have size $s \geq 2$, and that K^{α} and K^y have sizes $s - 1$. Then the availability of K^{x_1} or K^{x_2} makes K^{α} non-available and K^y available. In gate DUPLIC, we assume that the size of K^x is $s \geq 3$, the size of each K^{β_1} and K^{β_2} is $s - 2$, and the size of each K^{y_1} and K^{y_2} is $s - 1$. The availability of K^x is duplicated into K^{y_1} and K^{y_2}.[5]

By combining the logic gates and taking the vertex-cliques $K^x \in V_{i,j}$ as input, each $L_{i,j}$ is

[4]More precisely, this subgraph also includes the separating-clique K^{sx} (introduced at the end of the proof) of each vertex-clique K^x, with a similar conclusion.

[5]Note that gates OR and AND generalize from binary operators to multinary ones.

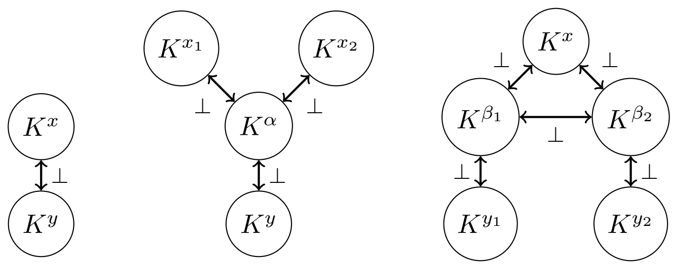

Figure 3: Logic gates NOT, OR and DUPLIC (input: K^x, output: K^y).

constructed to obtain formula $\left(\bigwedge_{K^x \in V_{i,j}} K^x\right) \bigvee \left(\bigwedge_{K^x \in V_{i,j}} \neg K^x\right)$, i.e. "all vertex-cliques $K^x \in V_{i,j}$ are available or none is". The validity of this formula is represented by the availability of the final output $K^o_{i,j}$ of $L_{i,j}$. As depicted in Figure 4, the output $K^o_{i,j}$ of each $L_{i,j}$ is then connected to a specific instance of Example 1, such that the core of game the $L_{i,j} \cup$ Example 1 exists if and only if the formula is valid.

Furthermore, to ensure that the main game is not altered by the logic games, each vertex-clique K^x is separated from $L_{i,j}$ by a double gate NOT, depicted in Figure 4. This double gate Not is composed of a separating-clique K^{sx} of size k' and an input-clique K^{ix} of size $(k'-1)$ (which is the actual input for the logic game $L_{i,j}$). Separating-clique K^{sx} is mutually neutral with K^x and K^{ix}, as well as with K^{φ}, and any vertex-clique K^w such that edge $(x, w) \in \mathcal{A}$ and its separating-clique K^{sw}. Thus, when K^x joins K^{φ}, both K^x and K^{sx} join it.

Finally, we explain the values of k', k'', and k^*. Since, in the instance of Example 1 associated with $L_{i,j}$, clique $K^{\mu}_{i,j}$ has only one friend (agent 1), we can set the size of output $K^o_{i,j}$ to 2. Recall that gate AND(K^{x_1}, K^{x_2}) is equal to NOT(OR(NOT(K^{x_1}), NOT(K^{x_2}))). It implies that between each input K^{ix} and the output $K^o_{i,j}$ of $L_{i,j}$, there are at most 1 gate Duplic, 2 gates OR and 2 gates NOT. Thus, the size of each input K^{ix} has to be at least 6. Due to separating-clique K^{sx}, we set the size of each vertex-clique K^x to 7, i.e., $k' = 7$.
Since each K^{sx} is mutually neutral with K^{φ}, a clique of size k in the original MinMaxClique instance implies a core stable coalition with $(2 \times k') \times k$ neutrals for K^{φ}. Thus, we set $k'' = 14k$.
Finally, excluding inhibitors, the maximal number of neutrals for agent v in $V_{i,j}$ is $7 + 14 \times (|\mathcal{V}| - 1) + 2$, that is, 7 neutrals from v's separating-clique, $14 \times (|\mathcal{V}| - 1)$ neutrals from the $(|\mathcal{V}| - 1)$ other vertex-cliques and their separating-cliques, and 2 neutrals from the fulcrum-clique. So we set $k^* = 14 \times (|\mathcal{V}| - 1) + 10$. ☐

4 Individual Stability

In this section, we investigate the existence of an individually stable coalition structure under both the original friends appreciation and friends appreciation with sociable agents.

4.1 Friends Appreciation with Indifferent Agents

Under the original friends appreciation, an individually stable outcome always exists in an HG/F without neutral agents [Dimitrov and Sung, 2004]. When adding neutral agents, although Ohta et al. [2017] argued that the core always exists, they did not address individual stability. We address it with Theorem 4 which proves that an individually stable outcome is no longer guaranteed.

Theorem 4. *In an HG/F, an individually stable coalition structure may not exist.*

To prove this theorem, we utilize the following example.

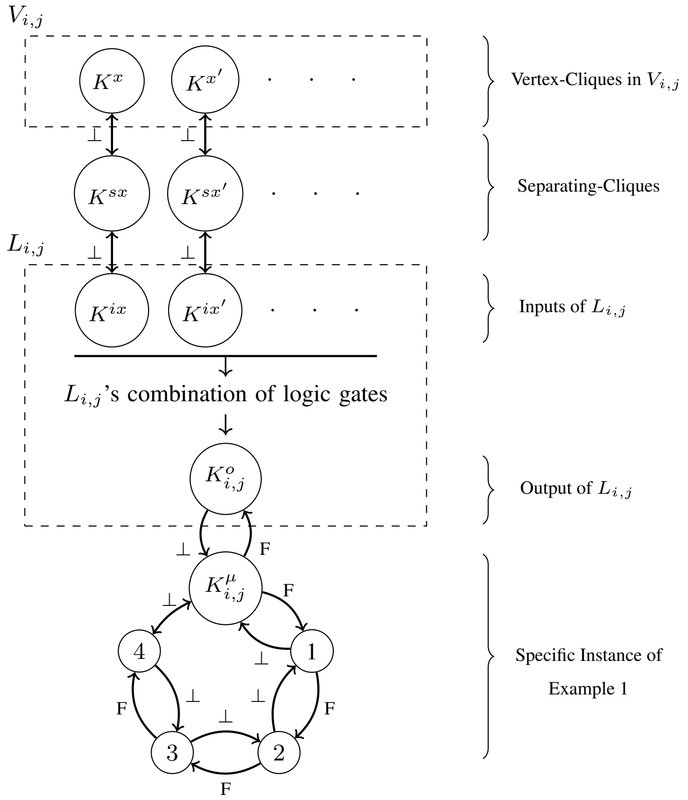

Figure 4: Connection of separating-cliques K^{sx} and logic game $L_{i,j}$.

Example 2. *Consider* 12 *agents* $\{0,\dots,11\}$*, divided into four groups:* $\{0,1,2\}$*,* $C_0 = \{3,4,5\}$*,* $C_1 = \{6,7,8\}$*, and* $C_2 = \{9,10,11\}$*, and the following preferences, illustrated in Figure 5 which is a partial representation of graph* $G_{EF\perp}$*.*
In this example and the following proof, when we write $[3]$*, we mean* $\pmod 3$*. First, for* $i \in \{0,1,2\}$*, preferences in groups* C_i *are defined as follows:*
• *agents* $3(i+1)+1$ *and* $3(i+1)+2$ *consider agent* $3(i+1)$ *a friend, and other relations within* C_i *are neutral;*
• *all agents in* C_i *consider agents* i *and* $i+1$ $[3]$ *neutral, agent* $i+2$ $[3]$ *an enemy, and agents from* $C_j, j \neq i$*, enemies.*
For $i \in \{0,1,2\}$*, agent* i*'s preferences are such that:*
• *agent* i *considers agent* $i+1$ $[3]$ *as friend, but agent* $i+1$ $[3]$ *regards agent* i *as neutral;*
• *agent* i *considers agent* $3(i+1)$ *as enemy and agents* $3(i+1)+1$ *and* $3(i+1)+2$ *as friend;*
• *agent* i *regards agent* $3((i+2\ [3])+1)$ *as neutral and agents* $3((i+2\ [3])+1)+1$ *and* $3((i+2\ [3])+1)+2$ *as friend;*
• *agent* i *is mutually enemy with any agent in* $C_{i+1[3]}$*.*

Proof. By contradiction, assume that an individually stable outcome π exists in Example 2.
We first focus on C_0. Notice that agent 3 has no friend in this game, implying that $\pi(3) \cap E_3 = \emptyset$; otherwise 3 deviates in a singleton coalition. Notice also that agents 3, 4, and 5 have identical preferences toward agents outside of C_0, and that 4 and 5 share a unique friend, that is agent 3. This implies that agents in C_0 are in the same coalition, which does not contain any enemy of 3.
Thus, by symmetry, for $i \in \{0,1,2\}$, agents in C_i belong to the same coalition, which does not

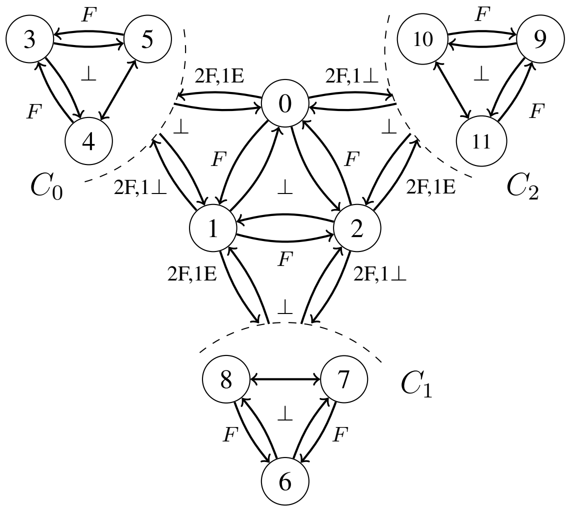

Figure 5: Partial representation of an HG/F with no individually stable outcome.

contain any enemy of agent $3(i+1)$. Furthermore, for $i \in \{0,1,2\}$, agent i belongs to the coalition of C_i or $C_{i+2[3]}$, where i has at least two friends. Thus, for $i \in \{0,1,2\}$, C_i is either alone, grouped with agent i, grouped with agent $i+1$ [3], or grouped with both.
Then, there are two cases to consider: (1) agents 0, 1, and 2 join three different coalitions, or (2) for $i \in \{0,1,2\}$, agents i and $i+1$ [3] belong to the same coalition. For $i \in \{0,1,2\}$:
Case (1): If agent i joins C_i, then i deviates toward coalition $\{i+2\ [3], C_{i+2[3]}\}$, where she has two friends and no enemy. However, if i joins $C_{i+2[3]}$, then i deviates toward coalition $\{i+1\ [3], C_i\}$, where she has three friends.
Case (2): Assume that agents i and $i+1$ [3] join the same coalition, i.e., $\pi(i) = \pi(i+1\ [3]) = \{i, i+1\ [3], C_i\}$. If agent $i+2$ [3] joins $C_{i+2[3]}$, then $i+2$ [3] deviates toward $C_{i+1[3]}$, where she has no enemy. However, if agent $i+2$ [3] joins $C_{i+1[3]}$, then $i+1$ [3] deviates toward coalition $\{i+2\ [3], C_{i+1[3]}\}$, where she has three friends.
As a result, no individually stable coalition structure exists. □

4.2 Friends Appreciation with Sociable Agents

In this subsection, we study individual stability in the presence of sociable agents. We prove that an individually stable coalition structure may not exist and that deciding its existence is NP-complete.

Theorem 5. *In an $HG/F+$, an individually stable coalition structure may not exist.*

Proof. The proof is based on the same example as in the proof of Theorem 1. Assume that an indiviudal stable coalition structure π exists in Example 1. Furthermore, assume that agents 5 and $5'$ belong to different coalitions.
Notice first that $\pi(4) \cap \{1,2\} = \emptyset$, since otherwise agent 4 deviates in a singleton coalition. It implies that agent 3 does not belong to $\pi(1)$, since otherwise 3 deviates in a singleton coalition.
Then, assume that agent 1 does not belong to $\pi(2)$. If $\pi(1) \cap \{5,5'\} = \emptyset$, then 5 deviates toward $\{1\}$. However, if $\pi(1) = \{1,5\}$ (resp. $\{1,5'\}$), then $5'$ (resp. 5) deviates toward $\{1,5\}$ (resp. $\{1,5'\}$).
Thus, 1 belongs to $\pi(2)$, which implies that $\pi(1) \cap \{5,5'\} = \emptyset$, since otherwise agent 2 deviates in a singleton coalition. Then, 3 does not belong to $\pi(5)$ or $\pi(5')$, since otherwise 5 or $5'$ deviates in singleton coalition. In other words, agents 5 and $5'$ are either both in singleton coalitions or just one of them is and the other belongs to $\pi(4)$. If 5 and $5'$ are both in singleton coalitions, then 5 deviates toward $\{5'\}$. Thus, $\pi(4) \cap \{5,5'\} \neq \emptyset$, but then 5 (resp. $5'$) deviates toward $\{4,5'\}$ (resp. $\{4,5\}$).

Therefore, 5 and $5'$ belong to the same coalition and we can apply a similar argument as in the proof of Theorem 1 to show that there is no individually stable coalition structure. □

Now we study the complexity of the existence of an individually stable outcome. First, given an HG/F+ and a coalition structure π, verifying whether π is individually stable is polynomial, since there is a polynomial number of individual deviations to consider. This result implies that the corresponding existence problem is in NP. Indeed, we show that it is NP-complete.

Theorem 6. *Problem* HG/F+/IS/EXIST *is NP-complete.*

Proof. As mentioned above, problem HG/F+/IS/EXIST is in NP. To prove NP-hardness, we reduce the NP-complete problem MAXCLIQUE to problem HG/F+/IS/EXIST.

Let graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ and threshold $k \in \mathbb{N}$ define an instance of MAXCLIQUE. We construct the corresponding instance of HG/F+/IS/EXIST with n vertex-agents in $V \equiv \mathcal{V}$ (modeling graph $\mathcal{G}$), n cliques of neutrals $K_1, \dots, K_n$, each of size k, and three agents $1', 2', 3'$; therefore the set of agents is $N = V \cup (K_i)_{i\in V} \cup \{1', 2', 3'\}$. The preferences are define as follows and are illustrated in Figure 6, which is a partial representation of graph $G_{EF\perp}$.
In set V, for each edge (i, j) in graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, we construct neutral arcs (i, j) and (j, i). Moreover, all agents in V consider agent $3'$ neutral, whereas agent $3'$ considers each of them friends. For all $i \in V$, all k agents in K_i are mutually neutral toward each other, mutually neutral with agent $i \in V$, and they consider agent $1'$ a friend, whereas agent $1'$ considers them neutrals. Furthermore, agent $1'$ (resp. $2'$) considers agent $2'$ (resp. $3'$) as friend, whereas agent $2'$ (resp. $3'$) considers agent $1'$ (resp. $2'$) as neutral. Finally, all other arcs are enemies; notice in particular that agents from two different cliques in $(K_i)_{i\in V}$ are enemies.
First, notice that the structure of the whole game is based on Example 1, with cliques $(K_i)_{i\in N}$ in

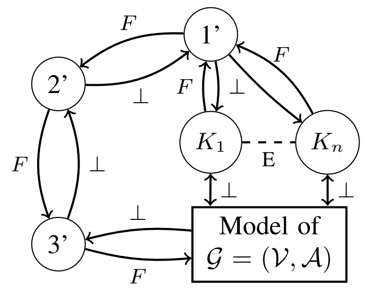

Figure 6: Instance of HG/F+/IS/EXIST from MAXCLIQUE.

place of agents 5 and $5'$, and agents in V (model of $\mathcal{G}$) embedded in place of agent 4. Thus, with a similar argument, we see that if there is no clique of size k in V, then no individually stable coalition structure exists. However, when a clique C of size k exists, the following coalition structure is individually stable: $\{\{C, 3'\}, \{1', 2'\}, (\{K_i, i\})_{i\notin C}, (\{K_i\})_{i\in C}\}$. Indeed, agent $i \in V$ in coalition $\{C, 3'\}$ has no interest in deviating toward K_i since the number of neutrals is the same. Since agent $2'$ cannot join agent $3'$ while $3'$ is not alone, the deviation cycle in Example 1 is interrupted. □

5 Discussion

In this section, we discuss the stability of coalition structures when considering introverted agents.
Friends appreciation with introverted agents When all agents are introverted, that is, they value neutral agents in their coalition negatively, enemies and neutral agents have similar (but not equivalent) impact on preferences. Thus, we obtain similar results as under friends appreciation without neutral agents, that is, core and individually stable coalition structures always exist. We prove this

result by showing that the strict core[6] always exists, with the same argument developed for the strict core under friends appreciation without neutral agents [Dimitrov et al., 2006].

Theorem 7. *Under friends appreciation with introverted agents (1) there always exists a strict core coalition structure and (2) it can be computed in polynomial time as the strongly connected components of graph* $G_F = (N, A_F)$.

By definition, a strict core stable coalition structure is also core and individually stable. Thus, core stable and individually stable coalition structures always exist and, moreover, can be computed in polynomial time.

Friends appreciation with sociable and introverted agents In practice, there exist both sociable and introverted agents at the same time. Clearly in this case, core and individual stability are not guaranteed, as the results obtained when only sociable agents exist extend.

6 Conclusion

We studied the impact of neutrals on stability in hedonic games under friends appreciation with sociable/introverted agents. With sociable agents, we provided counterexamples showing that both core stable and individually stable coalition structures might not exist. Then we examined the complexity of deciding the existence of such outcomes, proving that it is $\mathrm{NP}^{\mathrm{NP}}$-complete for core stability and NP-complete for individual stability. We also proved that an individually stable coalition structure might not exist under friends appreciation with indifferent agents, i.e., the original friends appreciation. Finally, we showed that with introverted agents a strict core/core/individually stable outcome always exists and can be computed in polynomial time.

In future works, we will explore graph structures or constraints on preferences (such as symmetry) required to guarantee core and individual stability under friends appreciation with social agents. We will also study the computational complexity regarding individual stability under friends appreciation with indifferent agents.

[6] A coalition structure $\pi \in C^N$ admits *weakly blocking coalition* $X \subseteq N$ iff for every $i \in X$, $X \succsim_i \pi(i)$, and there exists $j \in X$ such that $X \succ_j \pi(j)$. The *strict core* is the set of coalition structures that do not admit any weakly blocking coalition.

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